Existence of non-negative Solutions for cooperative elliptic Systems involving Schrödinger Operators in the whole Space

ABSTRACT. In this paper, we obtain some new results on the existence of non-negative solutions for systems of the form

\[-\Delta + q_i u_i = \mu_i m_i u_i + \sum_{j=1; j \neq i}^{n} a_{ij} u_j + f_i(x, u_1, \ldots, u_n) \text{ in } \mathbb{R}^N, \ i = 1, \ldots, n,\]

where each of the \( q_i \) are positive potentials satisfying \( \lim_{|x| \to +\infty} q_i(x) = +\infty \), each of the \( m_i \) are bounded positive weights, each of the \( a_{ij} \), \( i \neq j \), are bounded non-negative weights and each of the \( \mu_i \) are real parameters. Depending upon the hypotheses on \( f_i \), we obtain some new results by using sub- and super-solution methods and the Schauder Fixed Point Theorem.

1 Introduction

In this paper, we are interested in the existence of non-negative solutions of the following cooperative elliptic system

\[-\Delta + q_i u_i = \mu_i m_i u_i + \sum_{j=1; j \neq i}^{n} a_{ij} u_j + f_i(x, u_1, \ldots, u_n), \text{ in } \mathbb{R}^N, \ i = 1, \ldots, n. \tag{1.1}\]

We consider the following hypotheses for each \( i = 1, \ldots, n \) and \( j = 1, \ldots, n \)

- (h1) \( q_i \in L^2_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{loc}}(\mathbb{R}^N) \ (p > \frac{N}{2}) \) such that \( \lim_{|x| \to +\infty} q_i(x) = +\infty \) and \( q_i \geq \text{const} > 0 \).

- (h2) \( a_{ij} \in L^\infty(\mathbb{R}^N) \) and \( a_{ij} \geq 0 \) if \( i \neq j \).

- (h3) \( m_i \in L^\infty(\mathbb{R}^N) \) and there exists \( \alpha_i > 0 \) such that \( m_i(x) \geq \alpha_i > 0 \) for all \( x \in \mathbb{R}^N \).
Note that our system is cooperative since \( a_{ij} \geq 0 \) if \( i \neq j \). We will specify later the hypotheses on each \( f_i \) and we denote by \( \mu_i \) real parameters for \( i = 1, \ldots, n \).

The variational space is denoted by \( V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N) \) where for \( i = 1, \ldots, n \), \( V_{q_i}(\mathbb{R}^N) \) is the completion of \( D(\mathbb{R}^N) \), the set of \( C^\infty \) functions with compact support, under the norm

\[
\| u \|_{q_i} = \sqrt{\int_{\mathbb{R}^N} \left( |\nabla u|^2 + q_i u^2 \right)}.
\]

(1.2)

We recall that each of the embedding \( V_{q_i}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) is compact. We denote by

\[
\| u \|_{m_i} = \sqrt{\int_{\Omega} m_i u^2} \quad \text{for all} \quad u \in L^2(\Omega).
\]

According to the hypothesis \((h_3)\), \( \| \cdot \|_{m_i} \) is a norm in \( L^2(\mathbb{R}^N) \), equivalent to the usual norm so the embedding \( V_{q_i}(\mathbb{R}^N) \hookrightarrow (L^2(\mathbb{R}^N), \| \cdot \|_{m_i}) \) is still compact.

We denote by \( M_i \) the operator of multiplication by \( m_i \) in \( L^2(\mathbb{R}^N) \). The operator

\[
(-\Delta + q_i)^{-1} M_i : (L^2(\mathbb{R}^N), \| \cdot \|_{m_i}) \rightarrow (L^2(\mathbb{R}^N), \| \cdot \|_{m_i})
\]

is positive self-adjoint and compact. So its spectrum is discrete and consists of a positive sequence tending to 0. We denote by \( \lambda_i \) the first eigenvalue and by \( \phi_i \) the corresponding eigenfunction which satisfy

\[
(-\Delta + q_i)\phi_i = \lambda_i m_i \phi_i \quad \text{in} \quad \mathbb{R}^N, \quad \lambda_i > 0
\]

(1.3)

and \( \| \phi_i \|_{m_i} = 1 \). We recall that \( \lambda_i \) is simple and that \( \phi_i > 0 \) (see for examples [1, 2, 4, 5, 15, 16]). By the Courant-Fischer formulas, \( \lambda_i \) is given by

\[
\lambda_i = \inf \left\{ \frac{\int_{\mathbb{R}^N} \left( |\nabla \phi|^2 + q_i \phi^2 \right)}{\int_{\mathbb{R}^N} m_i \phi^2}, \phi \in D(\mathbb{R}^N) \right\}.
\]

(1.4)

The aim of this paper is to study the existence of non-negative solutions for the system (1.1). This extends earlier results obtained for the Laplacian operator in a bounded domain (see [12, 13]), for an operator of divergence form in a bounded domain (see [9]), for equations or systems involving Schrödinger operators \(-\Delta + q_i \) in \( \mathbb{R}^N \) (see [3, 6–8, 10, 11]).

Our paper is organized as follows. Section 2 provides some preliminaries and notations before stating our main result which is given in Section 3. In Section 4, we give some remarks on our hypotheses for a two-by-two system.
2 Preliminaries and Notations

2.1 Review of results for the scalar case \((i = 1)\)

We consider here the following equation, in a variational sense,

\[ (-\Delta + q)u = \lambda mu + g \text{ in } \mathbb{R}^N. \tag{e} \]

We assume the following: The potential \(q\) satisfies \((h_1)\), the weight \(m\) satisfies \((h_3)\), the constant \(\lambda\) is a real parameter and finally \(g \in L^2(\mathbb{R}^N)\). We denote by \((\lambda_{m,q}, \phi_{m,q})\) the eigenpair of eigenvalue and eigenfunction which satisfy

\[ (-\Delta + q)\phi_{m,q} = \lambda_{m,q} m \phi_{m,q}, \quad \lambda_{m,q} > 0, \phi_{m,q} > 0. \]

We recall the following results on the existence of solutions and the Maximum Principle.

**Theorem 2.1 (see [11]; Theorems 1.1 and 1.2)\)** Assume that \(\lambda < \lambda_{m,q}\). Then there exists a unique solution \(u \in V_q(\mathbb{R}^N)\) for the equation \((e)\). Moreover,

(i) the weak Maximum Principle holds: if \(g \geq 0\), then this solution \(u\) satisfies \(u \geq 0\),

(ii) the strong Maximum Principle holds too: if \(g \geq 0, g \neq 0\) then \(u > 0\).

2.2 Notations and Hypotheses

We recall that for each \(i = 1, \ldots, n\) the eigenpair \((\lambda_i, \phi_i)\) is defined by \((1.3)-(1.4)\). Let us denote by \(\Phi\) the vector defined by

\[ ^t\Phi = (\phi_1, \ldots, \phi_n). \tag{2.1} \]

We assume that for each \(i = 1, \ldots, n\), the nonlinear term \(f_i\) of the system \((1.1)\) satisfies the following hypotheses

\[(h_4)\] For each \(i = 1, \ldots, n\)

(i) \(f_i(x, k_1\phi_1, \ldots, k_n\phi_n) \in L^2(\mathbb{R}^N)\) for all positive numbers \(k_1, \ldots, k_n\).

(ii) \(0 \leq f_i(x, u_1, \ldots, u_n)\) for all \(u_1 \geq 0, \ldots, u_n \geq 0\).

(iii) For all \(0 \leq u_1 \leq v_1, \ldots, 0 \leq u_n \leq v_n\),

\[ 0 \leq f_i(x, u_1, \ldots, u_n) \leq f_i(x, v_1, \ldots, v_n). \]
(iv) For all positive real numbers $k_1, \ldots, k_n$,
\[
\frac{f_i(x, \eta k_1 \phi_1, \ldots, \eta k_n \phi_n)}{\eta \phi_i} \to 0 \text{ as } \eta \to +\infty, \text{ uniformly in } x.
\]

(v) $f_i$ is Lipschitz respect to $(u_1, \ldots, u_n)$ uniformly in $x$.

For instance, the function $f_i(x, u_1, \ldots, u_n) = \sqrt{u_1 + \cdots + u_n} 1_K$ (where $1_K$ denotes the indicator function on a compact $K \subset \mathbb{R}^N$) satisfies (h$_4$)(iv).

We denote by $L = (l_{ij})$ the following $n \times n$ matrix be defined by
\[
l_{ij} := \begin{cases} (\lambda_i - \mu_i)\alpha_i & \text{if } i = j, \\ -\|a_{ij}\|_{L^\infty(\mathbb{R}^N)} & \text{if } i \neq j, \end{cases}
\]
for all $i, j = 1, \ldots, n$. We also assume that the following hypothesis holds for some $\beta > 0$

(h$_5$) $(L - \beta I) \Phi \geq 0$

where $I$ is the $n \times n$ identity matrix. Here $(L - \beta I) \Phi \geq 0$ means that the entries of $(L - \beta I) \Phi$ are non-negative functions. Note that the hypothesis (h$_5$) forces that the coupling is very weak, i.e., with small coefficients $a_{ij}$ and with eigenfunctions $\phi_i$ which have the same behaviour at infinity: $\frac{\phi_i(x)}{\phi_j(x)}$ is bounded for all $i, j = 1, \ldots, n$ and all $x \in \mathbb{R}^N$. We can now develop our main result in the next section.

3 Existence of solutions

We begin stating our main result, obtained by considering a sub- and a super-solution of the system (1.1) and using the Schauder Fixed Point Theorem. We recall that $(v_1, \ldots, v_n)$ is a sub-solution (resp. a super-solution) of the system (1.1) if for each $i = 1, \ldots, n$, we have
\[
(-\Delta + q_i)v_i \leq \mu_i m_i v_i + \sum_{j=1, j \neq i}^{n} a_{ij} v_j + f_i(x, v_1, \ldots, v_n) \text{ in } \mathbb{R}^N
\]
(resp. $\geq$).

**Theorem 3.1**  Assume that the hypotheses (h$_1$)-(h$_5$) are satisfied. Then the system (1.1) has at least one non-negative solution in $V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$.

**Proof:** First, note that $u_0 = (0, \ldots, 0)$ is a sub-solution of the system (1.1). Then, by hypothesis (h$_5$), we have $(L - \beta I) \Phi \geq 0$ and so we get for each $i = 1, \ldots, n$,
\[
(\lambda_i - \mu_i) m_i \phi_i - \beta \phi_i - \sum_{j=1, j \neq i}^{n} a_{ij} \phi_j \geq 0.
\]
Since, by hypothesis $(h_4)(iv)$, we have for $\eta$ sufficiently large
\[
0 \leq \frac{f_i(x, \eta \phi_1, \ldots, \eta \phi_n)}{\eta \phi_i} \leq \beta,
\]
we can write for $\eta$ sufficiently large
\[
(\lambda_i - \mu_i)m_i \eta \phi_i - \sum_{j=1,j\neq i}^n a_{ij} \eta \phi_j \geq \eta \beta \phi_i \geq f_i(x, \eta \phi_1, \ldots, \eta \phi_n).
\]
Thus we have
\[
(\lambda_i - \mu_i)m_i \eta \phi_i - \sum_{j=1,j\neq i}^n a_{ij} \eta \phi_j \geq f_i(x, \eta \phi_1, \ldots, \eta \phi_n). \tag{3.2}
\]
Therefore $u^0 := \eta \Phi = (\eta \phi_1, \ldots, \eta \phi_n)$ is a super-solution of the system (1.1).

Now, we define the set $\sigma = [u_0, u^0]$. Let $\alpha$ be a positive real such that for all $i$, $\mu_i + \alpha > 0$. Let
\[
T : (L^2(\mathbb{R}^N))^n \to (L^2(\mathbb{R}^N))^n \quad (u_1, \ldots, u_n) = u \mapsto v = (v_1, \ldots, v_n)
\]
where for each $i = 1, \ldots, n$
\[
(-\Delta + q_i + \alpha m_i)v_i = (\mu_i + \alpha)m_i u_i + \sum_{j=1,j\neq i}^n a_{ij} u_j + f_i(x, u_1, \ldots, u_n) \text{ in } \mathbb{R}^N. \tag{3.3}
\]
Note that, by the scalar case, $T$ is well defined for all $u \in \sigma$.

As in [3], we prove now that $T(\sigma) \subset \sigma$. Let $u = (u_1, \ldots, u_n) \in \sigma$ and $T(u) = v = (v_1, \ldots, v_n)$. By the weak Maximum Principle for the scalar case, since the system (1.1) is a cooperative one and $\alpha > 0$, we get $v_i \geq 0$ for each $i = 1, \ldots, n$. Moreover, we have
\[
(-\Delta + q_i + \alpha m_i)(\eta \phi_i - v_i) = (\lambda_i + \alpha) \eta m_i \phi_i - (\mu_i + \alpha)m_i u_i
\]
\[
- \sum_{j=1,j\neq i}^n a_{ij} u_j - f_i(x, u_1, \ldots, u_n).
\]
By (3.2), we get
\[
(-\Delta + q_i + \alpha m_i)(\eta \phi_i - v_i)
\]
\[
\geq (\mu_i + \alpha)m_i (\eta \phi_i - u_i) + \sum_{j=1,j\neq i}^n a_{ij} (\eta \phi_j - u_j)
\]
\[
+ f_i(x, \eta \phi_1, \ldots, \eta \phi_n) - f_i(x, u_1, \ldots, u_n)
\]
\[
\geq 0.
\]
By the weak Maximum Principle for the scalar case, we obtain \( v_i \leq \eta \phi_i \) for each \( i = 1, \ldots, n \). Therefore \( T(\sigma) \subset \sigma \).

As in [3], we prove now that \( T \) is continuous and that \( T(\sigma) \) is compact. Let \((u_k)_k\), where \( u_k = (u_{1k}, \ldots, u_{nk}) \), be a sequence in \( \sigma \) and define \( T(u_k) = v_k \) where \( v_k = (v_{1k}, \ldots, v_{nk}) \). First, if \( (u_k) \) converges to \( u = (u_1, \ldots, u_n) \) for \( \| \cdot \|_{L^2(\Omega)^n} \), and if \( T(u) = v = (v_1, \ldots, v_n) \), from (3.3) we have for each \( i = 1, \ldots, n \) and for all \( k \)

\[
(-\Delta + q_i + \alpha m_i)(v_{ik} - v_i) = (\mu_i + \alpha) m_i (u_{ik} - u_i) + \sum_{j=1, j \neq i}^n a_{ij} (u_{jk} - u_j) + f_i(x, u_{1k}, \ldots, u_{nk}) - f_i(x, u_1, \ldots, u_n).
\] (3.4)

Multiplying (3.4) by \( v_{ik} - v_i \) and integrating over \( \mathbb{R}^N \), we get

\[
\|v_{ik} - v_i\|_{q_i + \alpha m_i}^2 = (\mu_i + \alpha) \int_{\mathbb{R}^N} m_i (u_{ik} - u_i) (v_{ik} - v_i) + \sum_{j=1, j \neq i}^n \int_{\mathbb{R}^N} a_{ij} (u_{jk} - u_j) (v_{ik} - v_i) + \int_{\mathbb{R}^N} (f_i(x, u_{1k}, \ldots, u_{nk}) - f_i(x, u_1, \ldots, u_n)) (v_{ik} - v_i).
\]

Using the hypothesis (h_4)(v) and the Cauchy-Schwartz inequality, since the coefficients \( m_i \) and \( a_{ij} \) are bounded, we deduce that there exists a constant \( C_1 > 0 \) such that

\[
\|v_{ik} - v_i\|_{q_i + \alpha m_i} \leq C_1 \sum_{j=1}^n \|u_{jk} - u_j\|_{L^2(\Omega)}.
\]

Therefore \( T \) is continuous. We prove now that \( T(\sigma) \) is compact. Multiplying (3.3) by \( v_{ik} \) we have also

\[
\|v_{ik}\|_{q_i + \alpha m_i}^2 = (\mu_i + \alpha) \int_{\mathbb{R}^N} m_i u_{ik} v_{ik} + \sum_{j=1, j \neq i}^n \int_{\mathbb{R}^N} a_{ij} u_{jk} v_{ik} + \int_{\mathbb{R}^N} f_i(x, u_{1k}, \ldots, u_{nk}) v_{ik}.
\]

Since the coefficients \( m_i \) and \( a_{ij} \) are bounded, then by the hypothesis (h_4), we see that \( f_i(x, u_{1k}, \ldots, u_{nk}) \) is bounded too and we can deduce the existence of a constant \( C_2 > 0 \) and of a constant \( C_3 > 0 \) such that

\[
\|v_{ik}\|_{q_i + \alpha m_i} \leq C_2 (\sum_{j=1}^n \|u_{jk}\|_{L^2(\mathbb{R}^N)} + C_3).
\]

But \( u \in \sigma \). Therefore the sequence \((v_k)_k\) is bounded in \( V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N) \) and since each of the embedding \( V_{q_i}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) is compact, we can find a subsequence of \((v_k)_k\) which is convergent in \((L^2(\mathbb{R}^N))^n\). Therefore \( T(\sigma) \) is compact.
By the Schauder Fixed Point Theorem, we deduce the existence of \( u \in \sigma \) such that \( T(u) = u \).

Clearly \( u \) is a non-negative solution of the system (1.1).

As in [14], we can relax the hypotheses on the increasing of each function \( f_i \) but assuming stronger hypotheses on the regularity of \( f_i \). For the next result, we will suppose that each function \( f_i \) of the system (1.1) satisfies the following hypothesis

\[(h'_4)\] (i) \( f_i(x, u_1, \ldots, \eta \phi_i, \ldots, u_n) \in L^2(\mathbb{R}^N) \) for all \( \eta > 0 \) and all \( 0 \leq u_1 \leq \eta \phi_1, \ldots, 0 \leq u_n \leq \eta \phi_n \).

(ii) \( 0 \leq f_i(x, u_1, \ldots, u_n) \) for all \( u_1 \geq 0, \ldots, u_n \geq 0 \).

(iii) \( f_i \) is of class \( C^1 \).

(iv) For all \( 0 \leq u_1 \leq \eta \phi_1, \ldots, 0 \leq u_n \leq \eta \phi_n \),

\[
\frac{f_i(x, u_1, \ldots, \eta \phi_i, \ldots, u_n)}{\eta \phi_i} \to 0 \text{ as } \eta \to +\infty, \text{ uniformly in } x.
\]

(v) \( f_i \) is Lipschitz respect to \((u_1, \ldots, u_n)\) uniformly in \( x \).

Following [14], we say that a couple \((u_0^1, \ldots, u_0^n) - (u_1^0, \ldots, u_n^0)\) is a sub–super-solution of the system (1.1) (Müller type conditions) if for each \( i = 1, \ldots, n \),

\[u_0^i \leq u_i^0\]

and moreover

\[
\begin{align*}
0 \geq (-\Delta + q_i)u_{0i} - \mu_i m_i u_{0i} & - \sum_{j=1; j \neq i}^n a_{ij} u_j - f_i(x, u_1, \ldots, u_{0i}, \ldots, u_n), \\
0 \leq (-\Delta + q_i)u_i^0 - \mu_i m_i u_i^0 & - \sum_{j=1; j \neq i}^n a_{ij} u_j - f_i(x, u_1, \ldots, u_i^0, \ldots, u_n),
\end{align*}
\]

(3.5)

for any \( u_j \in [u_{0j}, u_j^0] \).

It is clear that this definition is much more stringent than the natural definition (3.1) where (3.5) are only satisfied for \( u_j = u_{0j} \) (for the sub-solution) and for \( u_j = u_j^0 \) (for the super-solution). Note that if \( f_i \) is increasing, both definitions coincide.

**Theorem 3.2** Assume that the hypotheses \((h_1)-(h_3)\), \((h'_4)\) and \((h_5)\) are satisfied. Then the system (1.1) has at least one non-negative solution in \( V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N) \).

**Proof:** As in Theorem 3.1, we denote by \( u_0 = (0, \ldots, 0) \), \( \Phi = (\phi_1, \ldots, \phi_n) \) and by \( u^0 = \eta \Phi \) for \( \eta \) sufficiently large positive real defined later. First, we prove that \((u_0, u^0)\) is a couple of sub-super-solution in the sense of (3.5).
Indeed, proceeding as for Theorem 3.1, using hypotheses \((h'_4)\) and \((h_5)\) we have (for \(\eta\) sufficiently large)

\[
(\lambda_i - \mu_i) m_i \eta \phi_i \geq \sum_{j=1, j \neq i}^{n} a_{ij} \eta \phi_j + f_i(x, u_1, \ldots, \eta \phi_i, \ldots, u_n) \text{ for any } 0 \leq u_j \leq \eta \phi_j
\]

and therefore (since the system \((1.1)\) is cooperative)

\[
(\lambda_i - \mu_i) m_i \eta \phi_i \geq \sum_{j=1, j \neq i}^{n} a_{ij} u_j + f_i(x, u_1, \ldots, \eta \phi_i, \ldots, u_n) \text{ for any } 0 \leq u_j \leq \eta \phi_j.
\]

We define the set \(\sigma = [u_0, u^0]\). Let \(\alpha\) be a positive real such that for all \(i, \mu_i + \alpha > 0\). Let

\[
T_{\sigma}: \sigma \rightarrow (L^2(\mathbb{R}^N))^n
\]

\[
(u_1, \ldots, u_n) \mapsto v = (v_1, \ldots, v_n)
\]

where for each \(i = 1, \ldots, n\),

\[
(-\Delta + q_i + \alpha m_i + \rho m_i)v_i = (\mu_i + \alpha)m_i u_i + \sum_{j=1, j \neq i}^{n} a_{ij} u_j + f_i(x, u_1, \ldots, u_n) + \rho m_i u_i \text{ in } \mathbb{R}^N
\]

and where \(\rho > 0\) is a constant such that \(f_i(x, u_1, \ldots, u_n) + \rho m_i u_i\) is increasing in \(u_i\). We can find such \(\rho\) by hypotheses \((h_3)\) and \((h'_4)\) since the function \(f_i\) is \(C^1\). Still by the scalar case, the operator \(T_{\sigma}\) is well defined and proceeding as for Theorem 3.1, we can prove that \(T_{\sigma}\) is continuous and \(T_{\sigma}(\sigma)\) is compact.

Now we prove that \(T_{\sigma}(\sigma) \subset \sigma\). Let \(u = (u_1, \ldots, u_n) \in \sigma\) and \(T_{\sigma}(u) = v = (v_1, \ldots, v_n)\). Note by the scalar case \(v_i \geq 0\) for each \(i = 1, \ldots, n\). Moreover we have for each \(i = 1, \ldots, n\)

\[
(-\Delta + q_i + \alpha m_i + \rho m_i)(\eta \phi_i - v_i) \geq (\mu_i + \alpha)m_i(\eta \phi_i - u_i)
\]

\[
+ f_i(x, u_1, \ldots, \eta \phi_i, \ldots, u_n) + \rho m_i \eta \phi_i
\]

\[
- f_i(x, u_1, \ldots, u_n) - \rho m_i u_i
\]

\[
\geq 0.
\]

Since \(u_i \leq \eta \phi_i\) and \(w_i \mapsto f_i(x, u_1, \ldots, w_i, \ldots, u_n) + \rho m_i w_i\) is increasing, by the scalar case, we obtain \(v_i \leq \eta \phi_i\) for each \(i = 1, \ldots, n\). Therefore \(T_{\sigma}(\sigma) \subset \sigma\).

By the Schauder Fixed Point Theorem, we deduce the existence of at least one fixed point of \(T_{\sigma}\) or equivalently, one weak non-negative solution of the system \((1.1)\). \(\square\)
4 Study of a two-by-two system \((i = 2)\)

For a \(2 \times 2\) cooperative system with constant coefficients \(a, b, c, d\) and the same potential \(q\), if we rewrite the system (1.1) under the following form

\[
\begin{aligned}
(-\Delta + q)u &= au + bv + f(x, u, v) \quad \text{in } \mathbb{R}^N, \\
(-\Delta + q)v &= cu + dv + g(x, u, v) \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]  

(4.1)

the hypothesis \((h_5)\), \((L - \beta I)\Phi \geq 0\), means that

\((\lambda_q - a - b - \beta)\phi_q \geq 0\) and \((\lambda_q - c - d - \beta)\phi_q \geq 0\)

where \(\lambda_q\) is the principal eigenvalue associated with the eigenfunction \(\phi_q\) for the operator \(-\Delta + q\) (\(q\) being a potential satisfying the hypothesis \((h_1)\)). Since \(\phi_q > 0\), \((h_5)\) is equivalent, in this case, to \(\lambda_q \geq a + b + \beta\) and \(\lambda_q \geq c + d + \beta\). Therefore the hypothesis \((h_5)\) is stronger than the usual hypothesis in [3], [6–11], [12], [13] which is \(\lambda_q > a\), \(\lambda_q > d\) and \((\lambda_q - a)(\lambda_q - d) > bc\) or equivalently

\[
\begin{pmatrix}
\lambda_q - a & -b \\
-c & \lambda_q - d
\end{pmatrix}
\]  

(4.2)

is a non-singular M-matrix. However, for the nonlinear terms of the system (1.1), we consider here in Theorem 3.1 another class of functions \(f_i\) (denoted by \(f\) and \(g\) for \(n = 2\)) than the one used in [3] or [6–11] (where in these papers, each function \(f_i\) satisfies

\[0 \leq f_i(x, u_1, \ldots, u_n) \leq \theta_i\]

for all \(u_i \geq 0\) and with \(\theta_i\) a fixed function in \(L^2(\mathbb{R}^N)\)).

Moreover, for the system (4.1) when the function \(g\) depends only of \(u\) (\(g(u, v) := g(u)\)), using a decoupling method, we can prove the existence of a non-negative solution assuming that the nonlinear term \(f(x, u, v)\) satisfies \((h_4)\) and that the \(2 \times 2\) matrix defined by (4.2) is a non-singular M-matrix (which is the usual condition and a weaker hypothesis than \((h_5)\)).

So we now consider the following cooperative system

\[
\begin{aligned}
(-\Delta + q)u &= au + bv + f(x, u, v) \quad \text{in } \mathbb{R}^N, \\
(-\Delta + q)v &= cu + dv + g(x, u) \quad \text{in } \mathbb{R}^N.
\end{aligned}
\]  

(4.3)

**Theorem 4.1** Assume that the potential \(q\) satisfies the hypothesis \((h_1)\), the coefficients \(a, b, c, d\) are real parameters with \(b \geq 0\) and \(c \geq 0\), the function \(f\) satisfies the hypothesis \((h_4)\) respect to \(\phi_q\) the principal eigenfunction associated with \(\lambda_q\) the first eigenvalue of the operator \(-\Delta + q\). Assume also that the function \(g\) satisfies the following hypothesis
(h₆) (i) There exists a constant $K > 0$ such that $0 \leq g(u) \leq Ku$ for all $u \in L^2(\mathbb{R}^N)$, $u \geq 0$.

(ii) $g(u_1) \leq g(u_2)$ if $0 \leq u_1 \leq u_2$.

(iii) $g$ is Lipschitz respect to $u$ uniformly in $x$.

Assume that the $2 \times 2$ matrix $\Lambda$ be defined by

$$\Lambda = \begin{pmatrix} \lambda_q - a & -b \\ -(c + K) & \lambda_q - d \end{pmatrix}$$

is a non-singular M-matrix. Then the system $(4.3)$ has at least one non-negative solution $(u,v) \in (V_q(\mathbb{R}^N))^2$.

**Proof:** We use the decoupling method combined with the sub- and super-solution method. First, we recall that

$$(\Delta q + q)\phi_q = \lambda_q \phi_q \text{ in } \mathbb{R}^N$$  \hspace{1cm} (4.4)

with $\lambda_q > 0$ and $\phi_q > 0$. We proceed as in [1] and we define for $u \geq 0$ the continuous and compact operator

$$Bu := (-\Delta + q - d)^{-1}(cu + g(u)).$$  \hspace{1cm} (4.5)

Note that the operator $B$ is well defined since $d < \lambda_q$. Therefore $(u,v)$ is a solution of the system $(4.3)$ if and only if $v = Bu$ and

$$(-\Delta + q - a)u = bBu + f(x,u,Bu) \text{ in } \mathbb{R}^N.$$  \hspace{1cm} (4.6)

We denote by $u := 0$. By the weak Maximum Principle for the scalar case, since $c \geq 0$ and $g(u) \geq 0$, we have $Bu \geq 0$ and using the hypothesis $(h_4)$, we have also $f(x,u,Bu) \geq 0$. Therefore $u$ is a sub-solution of the equation $(4.6)$.

We construct now a super-solution of the equation $(4.6)$ of the form $\overline{u} = \eta \phi_q$ where $\eta$ will be a real positive parameter defined further on. Note that $\overline{u}$ is a super-solution of the equation if and only if

$$(\lambda_q - a)\eta \phi_q \geq bB\eta \phi_q + f(x,\eta \phi_q,B \eta \phi_q).$$  \hspace{1cm} (4.7)

We have

$$bB\eta \phi_q = \frac{bcn}{\lambda_q - d} \phi_q + b(-\Delta + q - d)^{-1}(g(\eta \phi_q)).$$

By the hypothesis upon $g$, we have $0 \leq g(\eta \phi_q) \leq K \eta \phi_q$. Still using the weak Maximum Principle for the scalar case, we deduce that

$$(-\Delta + q - d)^{-1}(g(\eta \phi_q)) \leq (-\Delta + q - d)^{-1}(K \eta \phi_q) = \frac{\eta K}{\lambda_q - d} \phi_q.$$
So we get
\[ bB\eta \phi_q \leq \frac{b\eta(c + K)}{\lambda_q - d} \phi_q. \]

Moreover, from the hypothesis which assures that $\Lambda$ is a non-singular M-matrix, we have
\[ \frac{(\lambda_q - a)(\lambda_q - d) - b(c + K)}{\lambda_q - d} > 0. \]

Since
\[ f(x, \eta \phi_q, B\eta \phi_q) \leq f(x, \eta \phi_q, \eta \frac{(c + K)}{\lambda_q - d} \phi_q), \]
by (h), we can choose a positive real $\eta$ sufficiently large such that
\[ 0 \leq \frac{f(x, \eta \phi_q, B\eta \phi_q)}{\eta \phi_q} < \frac{(\lambda_q - a)(\lambda_q - d) - b(c + K)}{\lambda_q - d}. \]

Therefore for $\eta$ sufficiently large and now fixed, we have
\[ bB\eta \phi_q + f(x, \eta \phi_q, B\eta \phi_q) \leq \eta \frac{b(c + K)}{\lambda_q - d} \phi_q + \frac{(\lambda_q - a)(\lambda_q - d) - b(c + K)}{\lambda_q - d} \eta \phi_q \]
and so (4.7) is satisfied or equivalently $\eta \phi_q$ is a super-solution of the equation (4.6).

Now we define the operator $T$ on $\sigma = [u, \bar{u}]$ by
\[ Tu := (-\Delta + q - a)^{-1}(bBu + f(x, u, Bu)). \]

Still again, the operator $T$ is well defined since $a < \lambda_q$, $Bu \in L^2(\mathbb{R}^N)$ and $f(x, u, Bu) \in L^2(\mathbb{R}^N)$ for all $u \in \sigma$. We prove that $T(\sigma) \subset \sigma$. Let $u \in \sigma$. Since $u \geq 0$ then $Bu \geq 0$ and so $f(x, u, Bu) \geq 0$ by the weak Maximum Principle for the scalar case. Therefore $Tu \geq 0$. We have from (4.7) and (4.8)
\[ (-\Delta + q - a)(Tu) = bBu + f(x, u, Bu) \]
and
\[ (-\Delta + q - a)\eta \phi_q = (\lambda_q - a)\eta \phi_q \geq bB\eta \phi_q + f(x, \eta \phi_q, B\eta \phi_q). \]

So we get
\[ (-\Delta + q - a)(\eta \phi_q - Tu) \geq b(B\eta \phi_q - Bu) + f(x, \eta \phi_q, B\eta \phi_q) - f(x, u, Bu) \text{ in } \mathbb{R}^N. \]

Moreover, since $g$ is an increasing function with respect to $u$, by the weak Maximum Principle for the scalar case, we deduce that $B$ is an increasing function with respect to $u$ too. Therefore, using (h) for $f$, we obtain
\[ (-\Delta + q - a)(\eta \phi_q - Tu) \geq 0. \]
The weak Maximum Principle allows us to conclude that $Tu \leq \eta \phi_q$ since $a < \lambda_q$. As for
Theorem 3.1, we can prove that $T$ is a continuous operator for the $L^2(\mathbb{R}^N)$-norm and by the
compact embedding $V_q(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ we get that $T(\sigma)$ is compact.

By the Schauder Fixed Point Theorem, we deduce the existence of $u_0 \in V_q(\mathbb{R}^N)$ such that

$$(-\Delta + q)u_0 = au_0 + Bu_0 + f(x, u_0, Bu_0) \text{ in } \mathbb{R}^N.$$  

Clearly, $(u_0, Bu_0)$ is a non-negative solution of the system (4.3).

Note that if we add an hypothesis on the nonlinear term $g$, then we can construct a sub-
solution of the equation (4.6) of the form $u = \epsilon \phi_q$ and consequently, we get the existence of
a positive solution of the system (4.3). This is the following result.

**Theorem 4.2** Assume that the potential $q$ satisfies the hypothesis $(h_1)$, the coefficients $a, b, c, d$ are real parameters with $b > 0$ and $c \geq 0$, the function $f$ satisfies the hypothesis
$(h_4)$ respect to $\phi_q$ the principal eigenfunction associated with $\lambda_q$ the first eigenvalue of the
operator $-\Delta + q$. Assume also that the function $g$ satisfies the hypothesis $(h_6)$ and the
following hypothesis

$$\lim_{s \to 0^+} \frac{g(s)}{s} \geq \frac{(\lambda_q - a)(\lambda_q - d) - bc}{b}.$$  

Assume that the $2 \times 2$ matrix $\Lambda$ be defined by

$$\Lambda = \begin{pmatrix} \lambda_q - a & -b \\ -(c + K) & \lambda_q - d \end{pmatrix}$$  

is a non-singular $M$-matrix. Then the system (4.3) has at least one positive solution $(u, v) \in (V_q(\mathbb{R}^N))^2$.

**Proof:** We proceed as for Theorem 4.1. We construct a sub-solution of the equation (4.6)
of the form $u = \epsilon \phi_q$ such that $u \leq s_0$ where $s_0$ is a positive real sufficiently small which satisfies

$$\frac{g(s)}{s} \geq \frac{(\lambda_q - a)(\lambda_q - d) - bc}{b}$$  

for all $0 < s \leq s_0$. This is possible due to the boundedness of the function $\phi_q$.

Indeed, since $0 < \epsilon \phi_q \leq s_0$, then

$$g(\epsilon \phi_q) \geq \frac{(\lambda_q - a)(\lambda_q - d) - bc}{b} \epsilon \phi_q.$$  

Thus, by the Maximum Principle for the scalar case, we have:

$$(-\Delta + q - d)^{-1}g(\epsilon \phi_q) \geq \frac{(\lambda_q - a)(\lambda_q - d) - bc}{b(\lambda_q - d)} \epsilon \phi_q.$$
and so
\[
bB(\epsilon \phi_q) \geq \frac{bc}{\lambda_q - d} \epsilon \phi_q + \frac{(\lambda_q - a)(\lambda_q - d) - bc}{(\lambda_q - d)} \epsilon \phi_q = (\lambda_q - a) \epsilon \phi_q.
\]
Since \( f(x, \epsilon \phi_q, B \epsilon \phi_q) \geq 0 \), we well deduce that \( u = \epsilon \phi_q \) is a sub-solution of the equation (4.6).

We can conclude as for Theorem 4.1 applying the Schauder Fixed Point Theorem for the operator \( T \) defined by (4.8) in the set \([\epsilon \phi_q, \eta \phi_q]\). We have just to verify that \( T([\epsilon \phi_q, \eta \phi_q]) \subset [\epsilon \phi_q, \eta \phi_q] \) i.e. if \( \epsilon \phi_q \leq u \leq \eta \phi_q \), then \( Tu \geq \epsilon \phi_q \). Indeed, from (4.6), since \( \epsilon \phi_q \) is a sub-solution of (4.6), we have:
\[
(-\Delta + q - a)(Tu - \epsilon \phi_q) \geq b(Bu - B(\epsilon \phi_q)) + f(x, u, Bu) - f(x, \epsilon \phi_q, B \epsilon \phi_q).
\]
By the Maximum Principle for the scalar case, since \( a < \lambda_q \), we get \( \epsilon \phi_q \leq Tu \).

We conclude giving a uniqueness result. As in [3], we add for that the following hypothesis

\((h_7)\) There exists a concave function \( H \) such that \( f(x, u, v) = b \frac{\partial H}{\partial u}(x, u, v) \) and \( g(x, u) = c \frac{\partial H}{\partial v}(x, u, v) \) for all \( u, v \).

Then, proceeding exactly as in [3], we have the following result.

**Theorem 4.3** Assume that the potential \( q \) satisfies the hypothesis \((h_1)\), the coefficients \( a, b, c, d \) are real parameters with \( b > 0 \) and \( c > 0 \), the function \( f \) satisfies the hypothesis \((h_4)\), the function \( g \) satisfies the hypothesis \((h_6)\). Assume that the \( 2 \times 2 \) matrix \( \Lambda \) be defined by
\[
\Lambda = \begin{pmatrix}
\lambda_q - a & -b \\
-(c + K) & \lambda_q - d
\end{pmatrix}
\]
is a non-singular M-matrix and the hypothesis \((h_7)\) is satisfied. Then the system (4.3) has a unique positive solution \((u, v) \in (V_q(\mathbb{R}^N))^2\).

**References**


Existence of non-negative Solutions for . . .

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