MA 1506 Mathematics II

Tutorial 1

First order differential equations

Groups: B03 & B08
January 25, 2012
Outline

First of all, HAPPY LUNAR NEW YEAR.

About me:

- My name? Well, just call me NGO (family name).
- I am a 4th year graduate student from Mathematics department (FoS). My office is located at S17.
- My current interests are mathematical physics and conformal geometry, such as,
  - The Einstein field equations in general relativity,
  - The Chern-Simons theories in quantum field theory.
- All emails to g0800876@nus are welcome.

About the tutorial:

- This is a 45-minute-tutorial, so please be punctual.
- Checking attendance: YES.
- My slot for consultation: 4-5pm, Wed @ E1-05-32.
- Slides will be uploaded to my blog at anhngq.wordpress.com.
Outline of Chapter 1 - Differential equations

- Definition of DEs:
  
  \[ y' = f(x, y), \quad \frac{dy}{dx} = g(x, y), \quad y'' = h(x, y), \ldots \]

  What differences between DEs and algebraic equations?
  What differences between DEs and PDEs?

- 1st order separable equations:
  - Definition: \( M(x) - N(y)y' = 0 \),
  - Reduction to separable form,
  - Linear change of variable to \( y' = f(ax + by + c) \).

- 1st order linear ODEs:
  - Definition: \( \frac{dy}{dx} + P(x)y = Q(x) \),
  - Methods of solving - integrating factor
    \[ R(x) = e^{\int P(s)ds}, \]
  - Reduction to linear form from \( y' + P(x)y = Q(x)y^n \) by setting \( y^{1-n} = z \).
Outline of Chapter 1 - Differential equations

- 2nd order linear ODEs:
  - Definition: \( y'' + p(x)y' + q(x)y = F(x) \), homogeneous and nonhomogeneous ODEs,
  - Methods of solving homogeneous ODEs with constant coefficients \( y'' + py' + qy = 0 \),
  - Methods of solving nonhomogeneous ODEs:
    - Finding \( y_h(x) \), a solution of the homogeneous equation,
    - Finding \( y_p(x) \), a particular of the given equation, by using the method of undetermined coefficients and the method of variation of parameters.

- EqWorld - The World of Mathematical Equations: By using
  http://eqworld.ipmnet.ru/en/solutions/ode.htm, you can find formulas for lots of DEs such as
  - First-Order Ordinary Differential Equations
  - Second-Order Linear Ordinary Differential Equations
  - Second-Order Nonlinear Ordinary Differential Equations
  - Higher-Order Linear Ordinary Differential Equations
Question 1: Separable equations

Recall that a 1st order DE is separable if it can be written in the form

\[ M(x) - N(y)y' = 0 \quad \text{or} \quad M(x)dx = N(y)dy. \]

(a) The equation \( x(x + 1)y' = 1 \) can be rewritten as

\[ y' = \frac{1}{x(x + 1)}. \]

By integrating w.r.t \( x \), there holds

\[ y = \int \frac{1}{x(x + 1)} \, dx, \]

i.e.,

\[ y = \ln \left| \frac{x}{x + 1} \right| + c. \]

Notice that

\[ \frac{1}{x(x + 1)} = \frac{1}{x} - \frac{1}{x + 1}. \]
Question 1: Separable equations

(b) The equation \((\sec x) y' = \cos(5x)\) is just

\[
y' = \frac{\cos(5x)}{\sec x} = \frac{\cos(5x)}{\frac{1}{\cos x}} = \cos x \cos(5x).
\]

Therefore,

\[
y = \int \cos x \cos(5x) \, dx = \frac{1}{2} \left( \frac{1}{6} \sin(6x) + \frac{1}{4} \sin(4x) \right) + c,
\]

where we have just used the following formula

\[2 \cos x \cos(5x) = \cos(x + 5x) + \cos(x - 5x).\]

(c) The equation \(y' = e^{x-3y}\) can be rewritten as \(e^{3y}y' = e^x\).

By integrating w.r.t. \(x\), we get that

\[
\frac{1}{3} e^{3y} = e^x + c.
\]
Question 1: Separable equations

(d) Obviously, the equation \((1 + y)y' + (1 - 2x)y^2 = 0\) always admits \(y \equiv 0\) as a solution. We know assume that \(y \neq 0\). In addition, we further assume that \(y \neq 0\) at any point. Otherwise, by the continuity of solution of ODEs, in our case, there holds \(y \equiv 0\). Notice that the equation is just

\[
\frac{1 + y}{y^2} y' = 2x - 1.
\]

By integrating w.r.t. \(x\), one gets

\[
\int \frac{1 + y}{y^2} dy = \int (2x - 1) dx.
\]

Thus,

\[-\frac{1}{y} + \ln |y| = x^2 - x + c.\]

Notice that there is one thing left in the previous argument, i.e., either \(y \equiv 0\) or \(y \neq 0\) at all points. A full proof will be put in my blog.
**Question 2**

The unknown - the temperature - $T(t)$ which follows some rule. Basically, we need to

- Write down an equation describing the situation.
- Show that $T_{\text{env}}$ is an obvious solution.

The equation describing the situation can be given as follows

$$\frac{dT}{dt} = -k(T - T_{\text{env}})$$

for some constant $k > 0$ (since $T$ decreases because $T > T_{\text{env}}$). The sign of $k$ plays some role as you can see from my blog. This ODE can be solved to get

$$\ln |T - T_{\text{env}}| = -kt + c.$$
Question 3: Virga

Let’s denote by $V$ and $A$ the volume and the surface area of raindrops. Keep in mind that both $V$ and $A$ are functions of time $t$. There are two conditions:

\[
\frac{\text{the volume } V}{3/2 \text{ power of the surface area } A} = \text{constant}_1
\]

and

\[
\frac{\text{the rate of reduction of the volume } V}{\text{the surface area } A} = \text{constant}_2.
\]

Mathematically, there exist two positive constants $a$ and $b$ such that

\[
V = aA^{3/2}, \quad \frac{dV}{dt} = -bA.
\]

Notice that since $V$ decreases, $\frac{dV}{dt} < 0$. Therefore, there holds

\[
\frac{A'}{\sqrt{A}} = -\frac{2b}{3a}, \quad \text{or} \quad \frac{dA}{\sqrt{A}} = -\frac{2b}{3a}dt.
\]
Question 3: Virga

By integrating w.r.t. \( t \), one gets

\[ 2\sqrt{A} = -\frac{2b}{3a}t + c. \]

Suppose that at the time \( t = 0 \), the (initial) surface area is \( A_0 \), then we have

\[ 2\sqrt{A_0} = c, \quad 2\sqrt{0} = -\frac{2b}{3a}t + c. \]

By solving, we obtain \( t = \frac{3a\sqrt{A_0}}{b} \). If \( \frac{dV}{dt} = -bA^2 \), then we get

\[ \frac{A'}{A^{\frac{3}{2}}} = -\frac{2b}{3a}. \]

By integrating both sides w.r.t. \( t \), one arrives at

\[ -\frac{2}{\sqrt{A}} = -\frac{2b}{3a}t + c. \]

From here, one easily see that we cannot put \( A = 0 \).
Question 4: Trajectories of moths

Since the moon is too far way, the moth tends to fly straightforward (light beams from the moon are parallel, why?).

If there is a candle, the trajectory of the moth could be changed. More precise, it could be a curve.
Question 4: Trajectories of moths

Let us denote by $\psi$ the angle between the velocity vector and the radius vector (the direction vector). It is known that $\psi$ is fixed at all time.

In polar coordinates $(r, \theta)$ with the origin located at the candle, there holds

$$\tan \psi = r \frac{d\theta}{dr}, \quad \text{or} \quad \frac{dr}{r} = \frac{d\theta}{\tan \psi}.$$ 

For a proof, see wiki. By solving, one gets that

$$r = R e^{\frac{\theta}{\tan \psi}},$$

where $R$ is initial distance to the candle corresponding to $\theta = 0$.

Since the location of the moth depends on the distance function $r$ which obviously depends on the sign of $\tan \psi$, the asymptotic behavior of the moth therefore depends on $\psi$. 
**Question 4: Trajectories of moths**

In order to fully understand the solution, we should consider the following picture regarding to how big the angle $\psi$ is.

The final answer can be summarized as follows

- If $\psi > 90^0$, the moth will spiral into the candle.
- If $\psi = 90^0$, the moth will fly along a circle until it drops dead.
- If $\psi < 90^0$, the moth will spiral outward.
**Question 5: Using a change of variable**

As can easily be seen, both

\[
y' = \frac{1 - 2y - 4x}{1 + y + 2x}, \quad y' = \left(\frac{x + y + 1}{x + y + 3}\right)^2
\]

cannot be solved using the method used in Q1. Fortunately, there is a clue; there are some repeated expressions.

\[
y' = \frac{1 - 2(y + 2x)}{1 + y + 2x}, \quad y' = \left(\frac{x + y + 1}{x + y + 3}\right)^2.
\]

By denoting these repeated expressions, one can solve these equations to get

\[
y + 2x + \frac{1}{2}(y + 2x)^2 = 3x + c
\]

and

\[
x + y + \ln |(x + y)^2 + 4(x + y) + 5| = 2x + c.
\]